Optimization and physics:

the hardness of a problem and annealing strategies to solve it

JQI Summer School

Lecture on July 27th, 2018

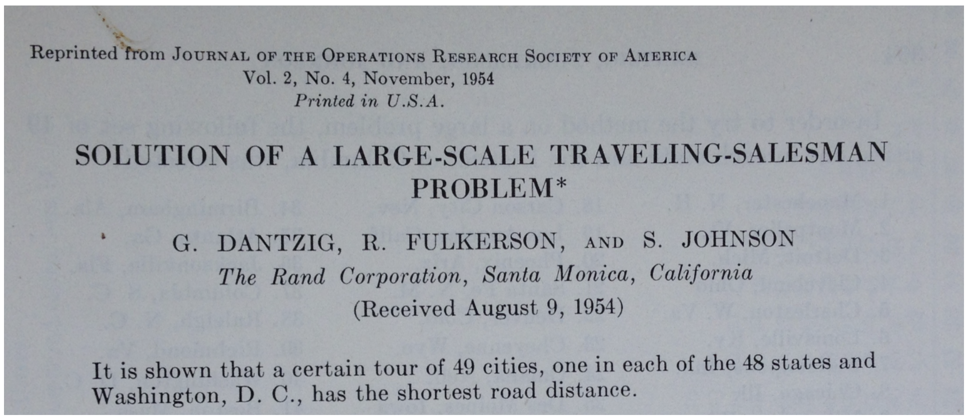
Tobias Grass (JQI)

Vacations coming :-)

In 1954:

Write a breakthrough paper!

(linear programming algorithm)



Reprinted from Journal of the Operations Research Society of America
Vol. 2, No. 4, November, 1954
Printed in U.S.A.

SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, and S. JOHNSON
The Rand Corporation, Santa Monica, California

(Received August 9, 1954)

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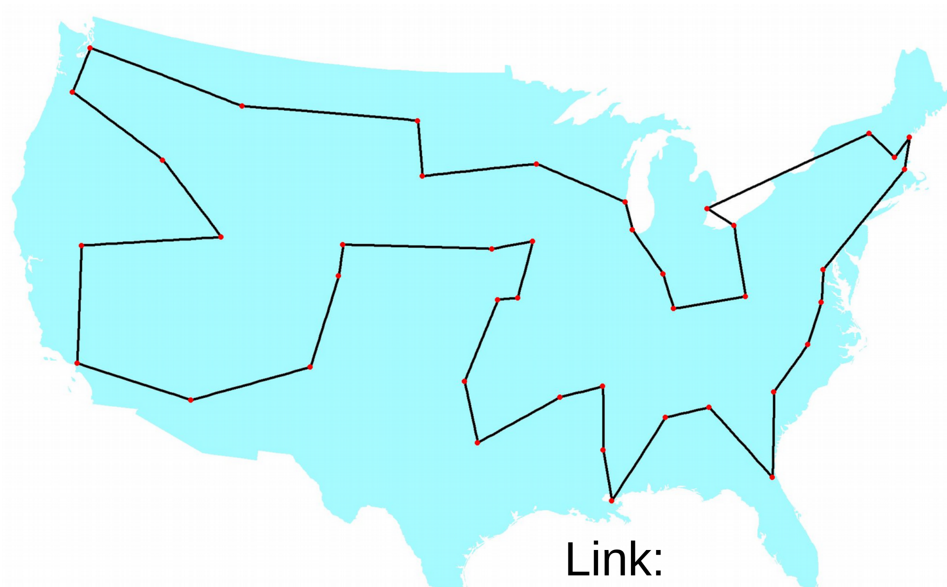

Link:
http://www.math.uwaterloo.ca/tsp/

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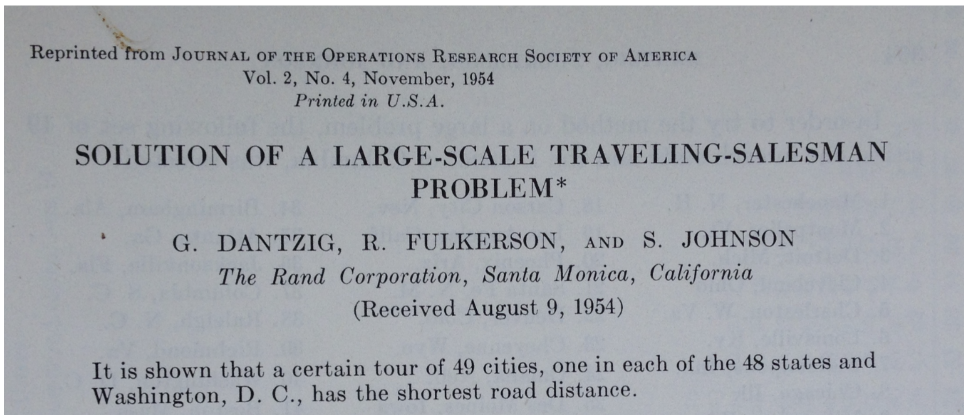

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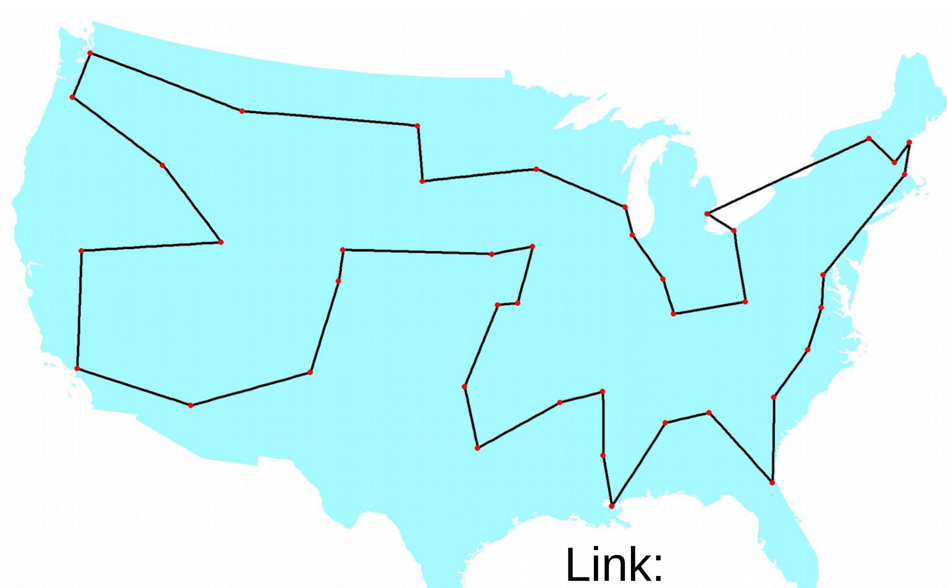
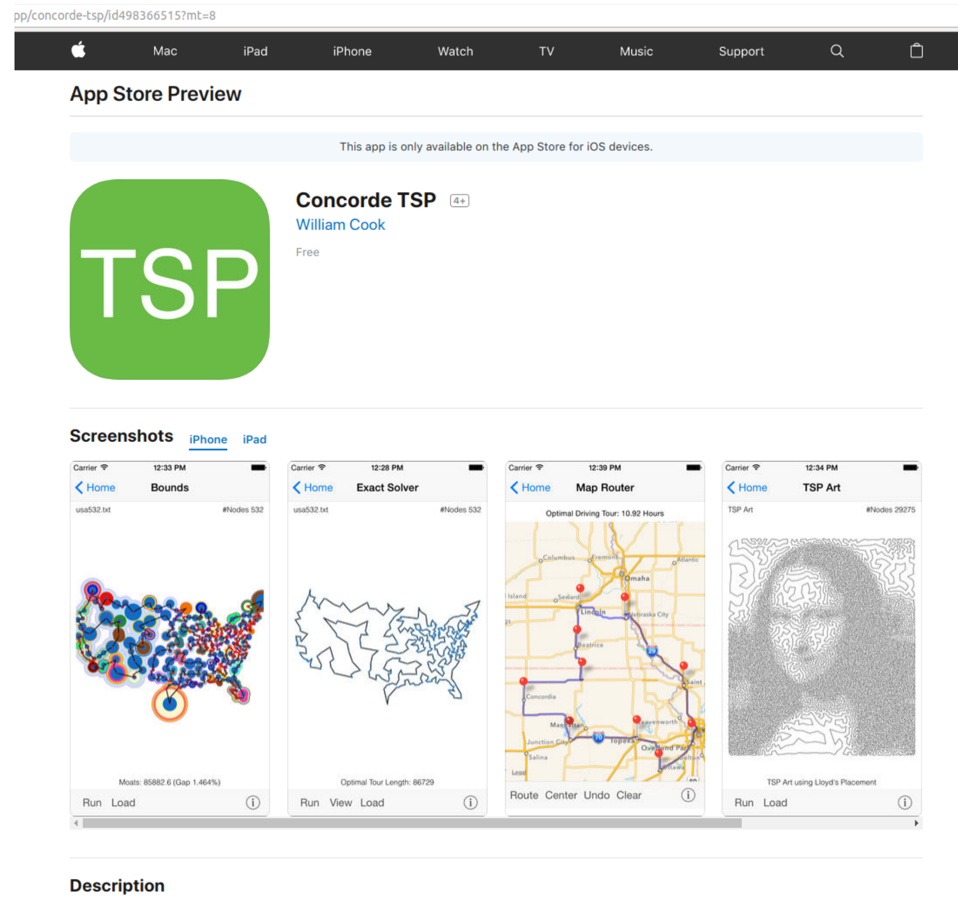


In 2018:

Ask Siri!



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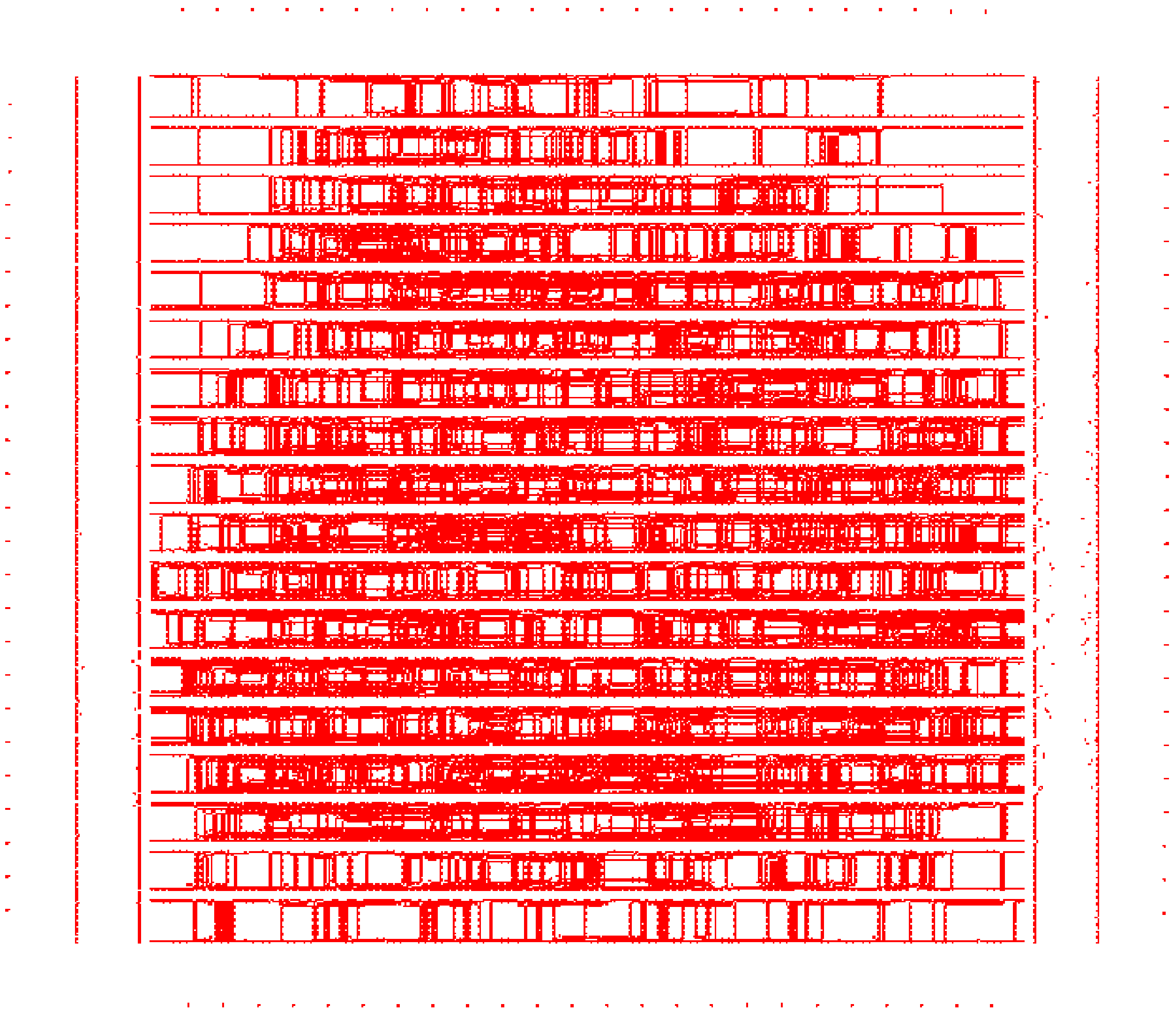
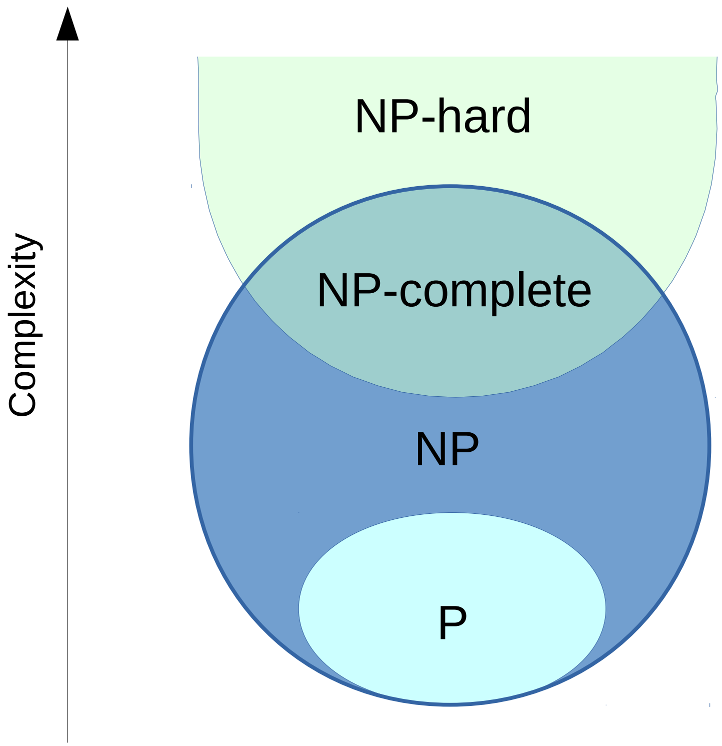
Problem solved? - Not really!

Even the best algorithms fail (i.e. take too long) when the problem size gets bigger

World record: 85,900 connections
Took 136 cpu-years to calculate
Design of computer chip





NP-hard: Problems at least as hard as NP-complete problems, but not necessarily in NP

NP-complete: “Hardest” problems in NP (to which any NP problem can be mapped in polynomial time)

NP: Decision problems whose positive answer can be *verified* on a deterministic computer in polynomial time,
that is equivalent to,
decision problems which can be *solved* on a **non-deterministic** computer in polynomial time.

P: Decision problems solvable on a deterministic computer in polynomial time

Open problem: $NP \supsetneq P$ or $NP = P$?

(Note the 1 million dollar reward for a proof!)

Examples for NP-hard problems:

- Traveling salesperson
- Number partitioning
- Exact cover
- Spin glasses

...


Journal of Physics A: Mathematical and General

Journal of Physics A: Mathematical and General > Volume 15 > Number 10

On the computational complexity of Ising spin glass models

F Barahona

Show affiliations

F Barahona 1982 *J. Phys. A: Math. Gen.* **15** 3241. doi:10.1088/0305-4470/15/10/028

1) Simulated annealing:

- Cooling a problem to its solution
- Example: traveling salesperson problem

2) Quantum annealing:

- Quantum time evolution to the solution
- Examples: Spin models, exact Cover
- Adiabatic theorem, limitations, and workarounds
- Physical implementations (D-Wave, ions)

3) Phases of a computational problem:

Statistical physics analysis applied to the number partitioning problem

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Thermodynamic sampling (Metropolis algorithm)

Statistical physics:

- Macroscopic behavior can be understood without knowing microscopic state
- Likelihood of a microscopic state *j* is controlled by Boltzmann factors: $p_j = \mathrm{e}^{-\frac{E_j}{k_\mathrm{B}T}}$
- Thermal averages: $\langle O \rangle_T = \frac{1}{\mathcal{Z}} \sum_j \mathrm{e}^{-\frac{E_j}{k_\mathrm{B}T}} \langle j|O|j\rangle$

Statistical physics:

- Macroscopic behavior does not depend on the microscopic state
- Likelihood of a microscopic state *j* is controlled by Boltzmann factors: $p_j = \mathrm{e}^{-\frac{E_j}{k_\mathrm{B}T}}$
- Thermal averages: $\langle O \rangle_T = \frac{1}{\mathcal{Z}} \sum_j \mathrm{e}^{-\frac{E_j}{k_\mathrm{B}T}} \langle j | O | j \rangle$

Metropolis sampling to evaluate average:

(1) Start with a random microscopic state *j*, and evaluate its energy E_j

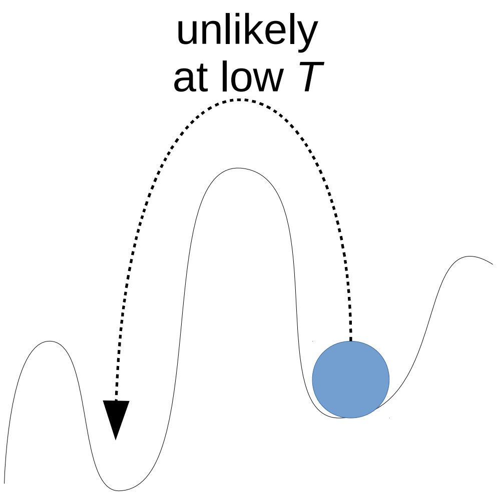
(2) Follow some (well chosen) rules to modify the state, leading to a new state *k* with energy E_k

(3) If $E_k \leq E_j$: Always accept new configuration: $p_{j\to k} = 1$

Else: Accept with probability $p_{j\to k} = \mathrm{e}^{-\frac{(E_k - E_j)}{k_\mathrm{B}T}}$

✔ Detailed balance: $p_j \; p_{j\to k} = p_k \; p_{k \to j}$

✔ Ergodicity: path between any pair of configurations?



Problem: Slow dynamics at low temperature
Algorithm remains in metastable solutions for a long time

Solution: Annealing → decrease temperature slowly!

Intuition:

- Heating and slow cooling can reduce defects in a material

Implementation:

- Use Metropolis algorithm for sampling in configuration space
- Energy ↔ Cost function
- Temperature as control parameter:
  - High temperature: explore gross “energy” landscape
  - Lower temperature: force the algorithm into good solution

13 May 1983, Volume 220, Number 4598 SCIENCE

Optimization by Simulated Annealing

S. Kirkpatrick, C. D. Gelatt, Jr., M. P. Vecchi

Annealing, as implemented by the Metropolis procedure, differs from iterative improvement in that the procedure need not get stuck since transitions out of a local optimum are always possible at nonzero temperature. A second and more important feature is that a sort of adaptive divide-and-conquer occurs. Gross features of the eventual state of the system appear at higher temperatures; fine details develop at lower temperatures. This will be discussed with specific examples.

Simulated annealing in the traveling salesperson problem

Example: 400 cities in 9 clusters

- High-T: find an efficient way to connect clusters
- Low-T: optimize connection within each cluster

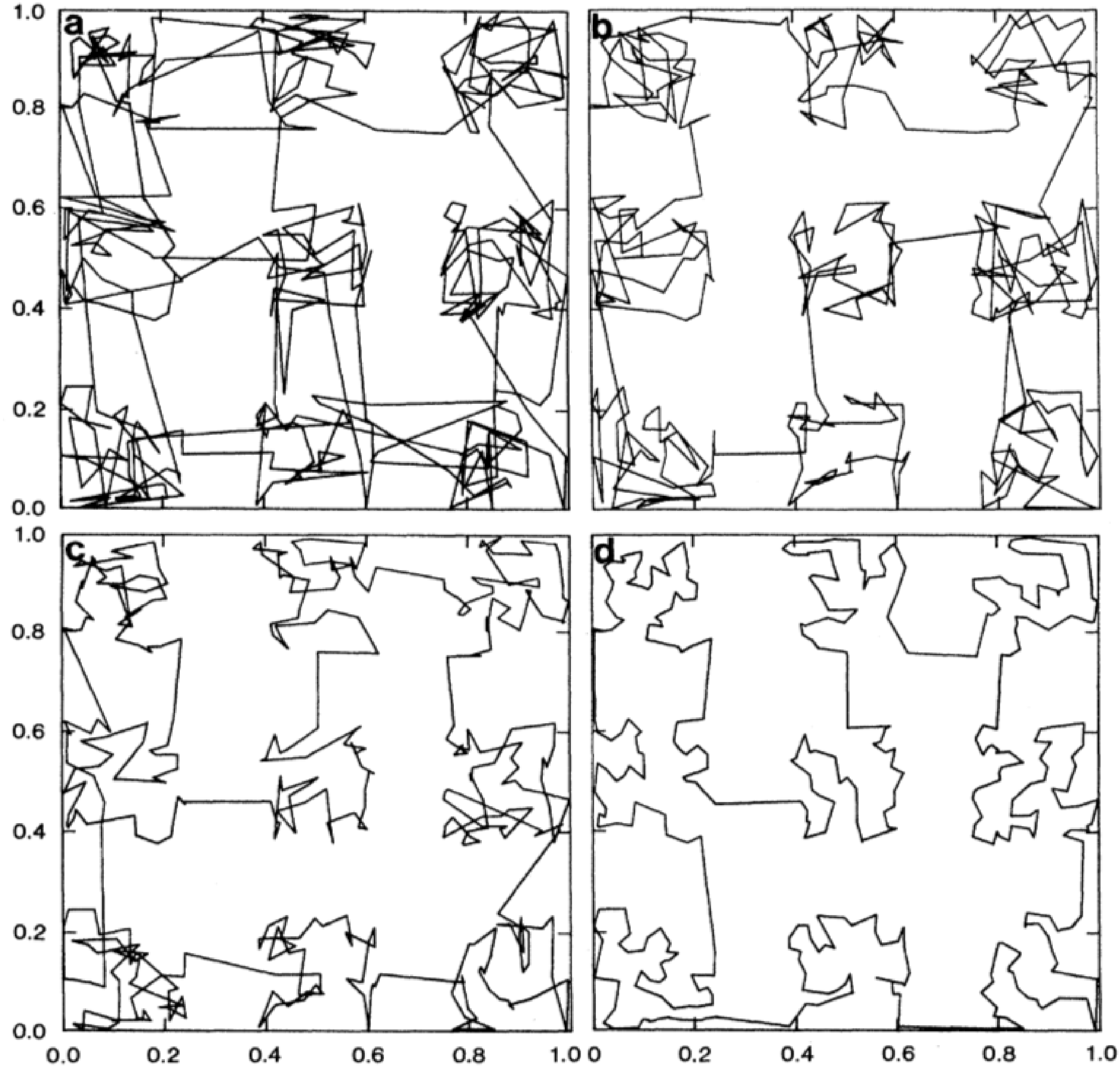


Fig. 9. Results at four temperatures for a clustered 400-city traveling salesman problem. The points are uniformly distributed in nine regions. (a) $T = 1.2$, $\alpha = 2.0567$; (b) $T = 0.8$, $\alpha = 1.515$; (c) $T = 0.4$, $\alpha = 1.055$; (d) $T = 0.0$, $\alpha = 0.7839$.

Simulated annealing in the traveling salesperson problem

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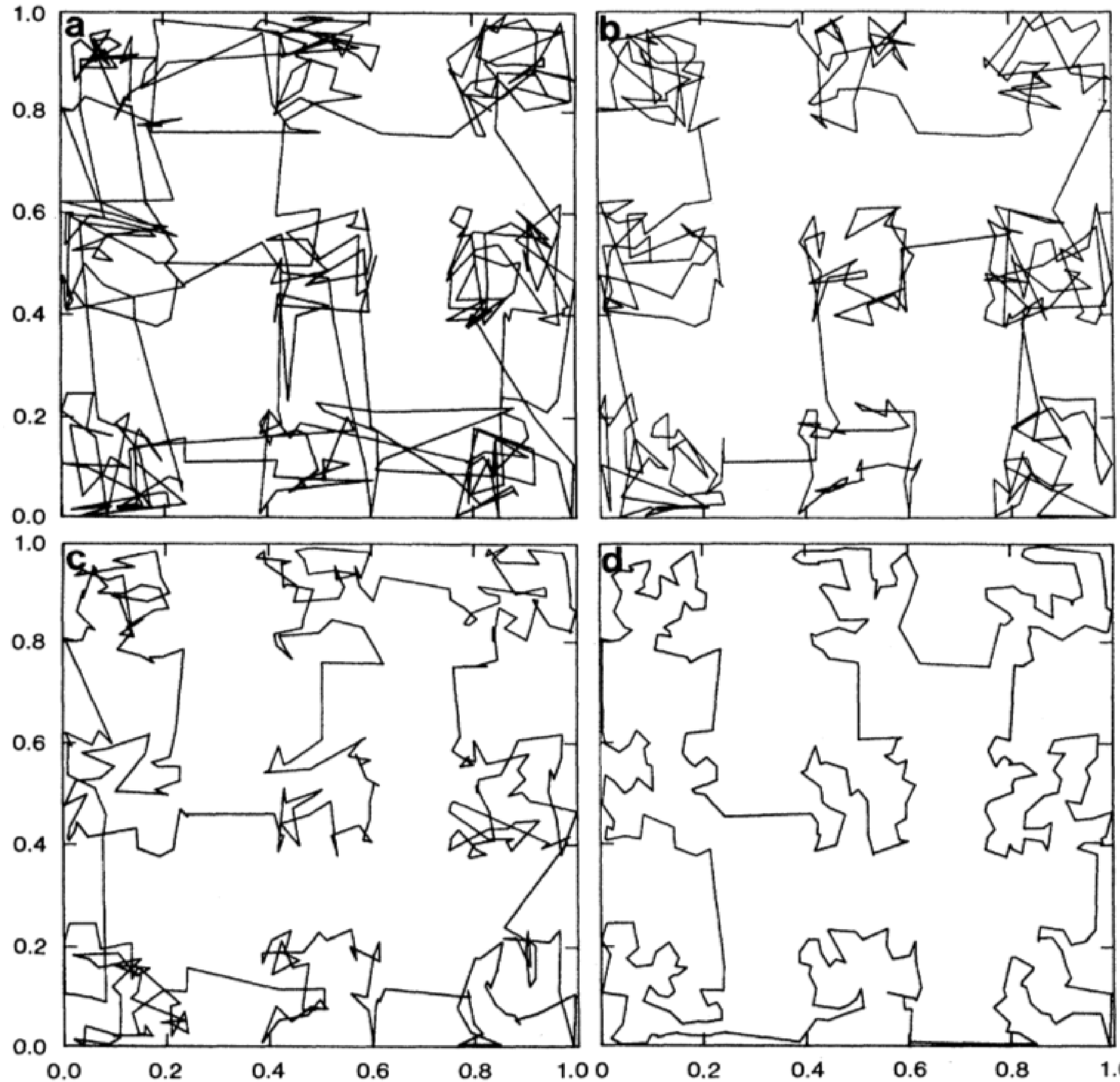
Simulated annealing typically performs well in finding a good solution, but it usually fails to give the best solution.



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From simulated to quantum annealing

PHYSICAL REVIEW E VOLUME 58, NUMBER 5 NOVEMBER 1998

Quantum annealing in the transverse Ising model

Tadashi Kadowaki and Hidetoshi Nishimori
Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan
(Received 30 April 1998)

	Simulated annealing (SA)	Quantum annealing (QA)
Dynamics	Master equation $$\frac{\mathrm{d}}{\mathrm{d}t}P_i(t) = \mathcal{L}_{ij}(t)P_j(t)$$	Schroedinger equation $$i\hbar\partial_t\Psi(t) = \mathcal{H}(t)\Psi(t)$$
Driving force	Thermal fluctuations	Quantum fluctuations
Control parameter	Temperature *(from equilibrium at high T to equilibrium at zero T)*	Quantum field strength *(from ground state at strong field to ground state at no field)*

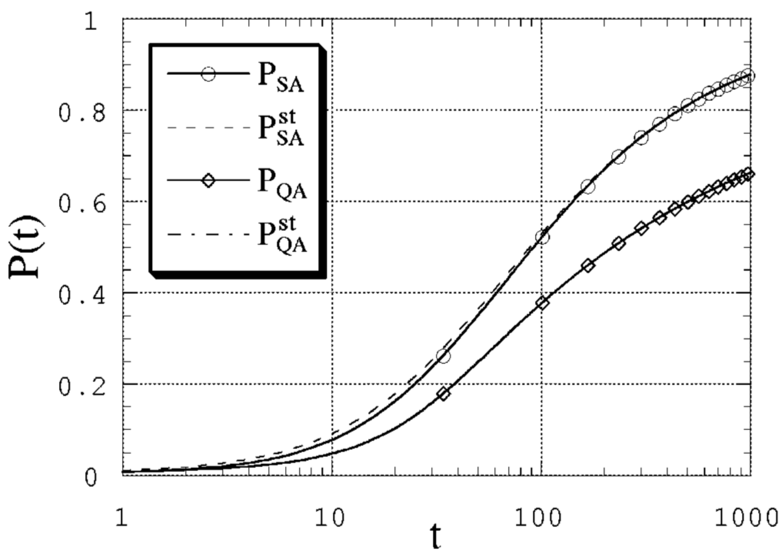
- Cost function given by the Hamiltonian of a a (classical) Ising model: $$H = -\sum_{i,j} J_{ij}\sigma_z^i\sigma_z^j - \sum h_i\sigma_z^i$$
- Transverse field to control quantum fluctuations: $$\mathcal{H}(t) = H + \Gamma(t)\sum_i \sigma_x^i$$

Which dynamics, SA vs QA, comes closer/faster to the ground state ?

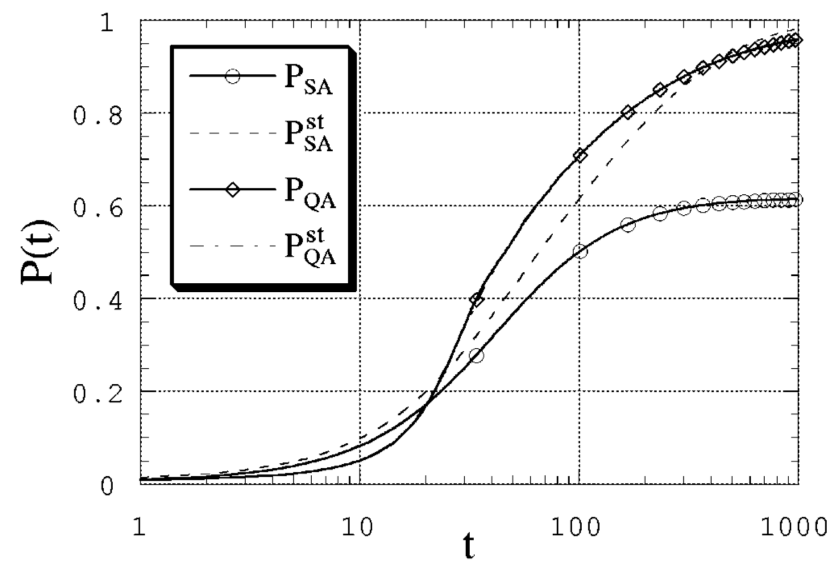
(I) Ferromagnetic model

(II) Spin glass model

$$\Gamma(t) = T(t) \sim 1/\ln(t+1)$$



$$\Gamma(t) = T(t) \sim 1/t$$



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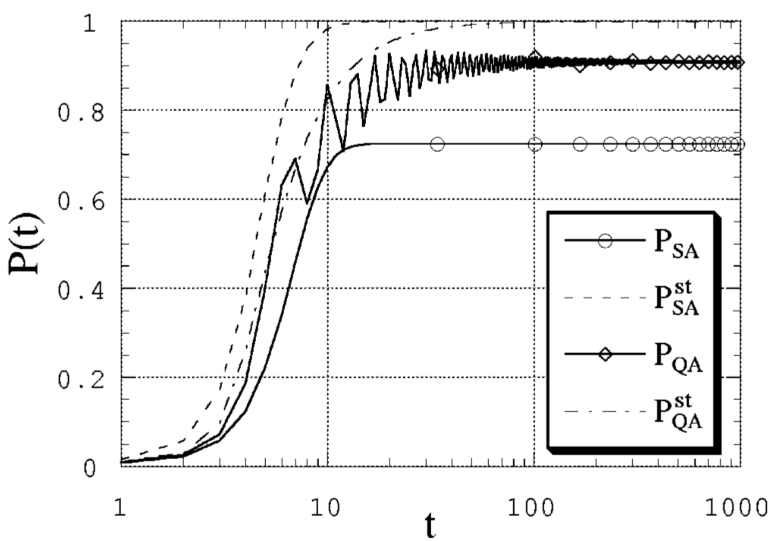


Results:

- Slow decay: Both SA and QA reproduce stationary states
- Fast decay: QA gets closer to ground state
- QA works particularly well in glassy system

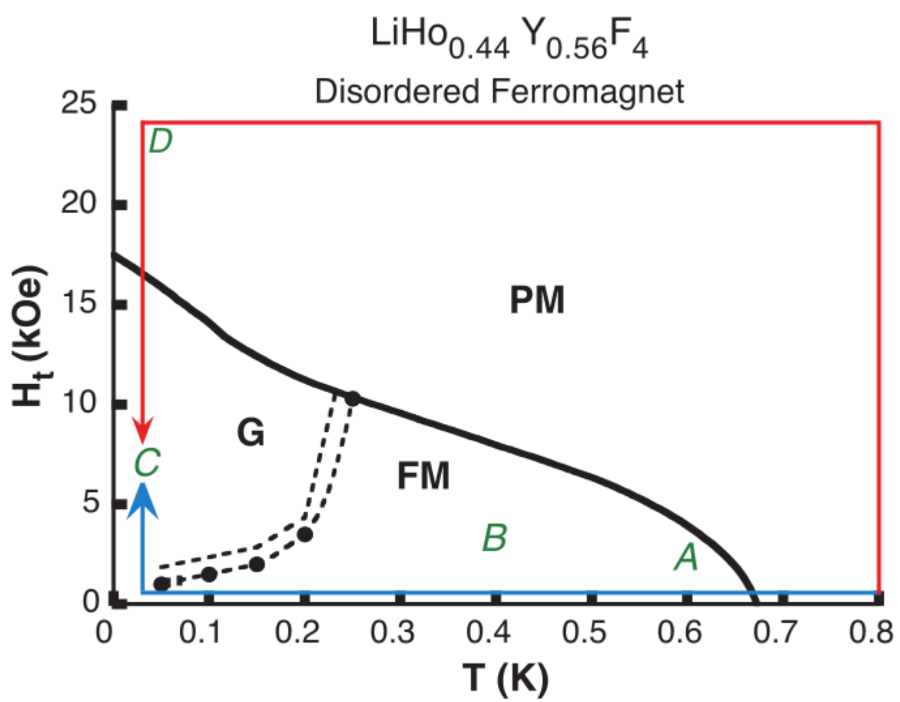
Disordered quantum magnet

Doped material with randomly distributed spins:

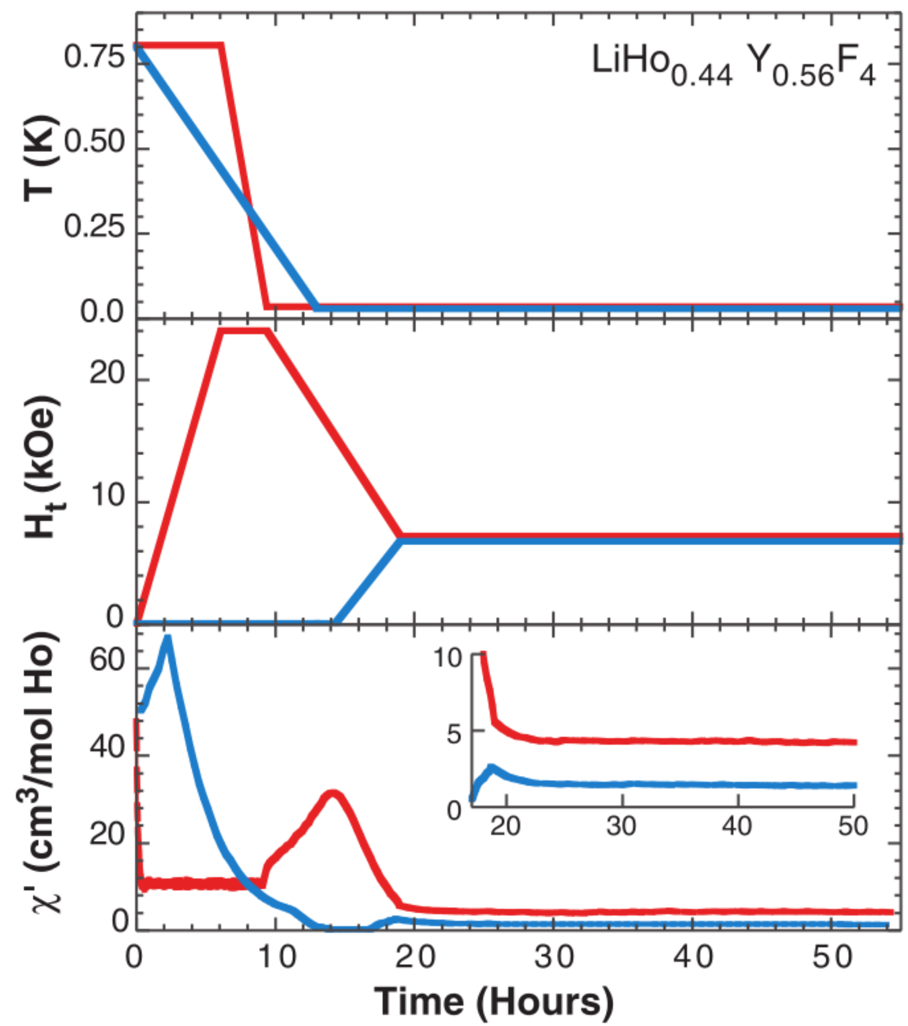
$$H_J = -\sum_{i,j} J_{ij}\sigma_z^i\sigma_z^j$$

Transverse magnetic field term: $H_t = -\Gamma\sum_i \sigma_x^i$

Phase diagram:

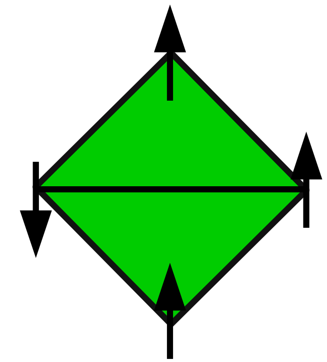


Cooling schedule and ac susceptibility (at 15Hz) as a function of time:



J. Brooke, D. Bitko, T. F. Rosenbaum, G. Aeppli
Science 284, 779 (1999)

Relation to computer science:



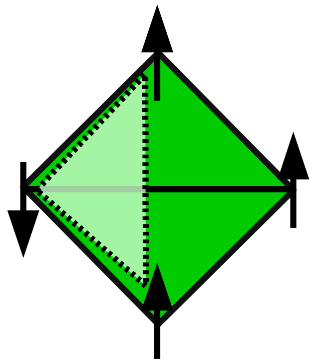
Can quantum annealing solve an NP-hard computational problem?

Example in early proposal: "Exact Cover 3"

- Given N bits $\{z_1,\ldots,z_N\}$ and M clauses
  Each clause selects three bits $\{z_i,z_j,z_k\}$, demanding $z_i+z_j+z_k=2$
- Is there a bit assignment which satisfies all clauses simultaneously?

E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, D. Preda
Science 292, 472 (2001)

Back to computation:



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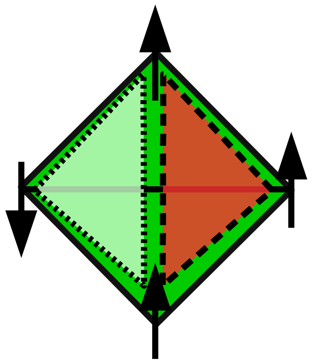
Quantum annealing for Exact Cover Problem



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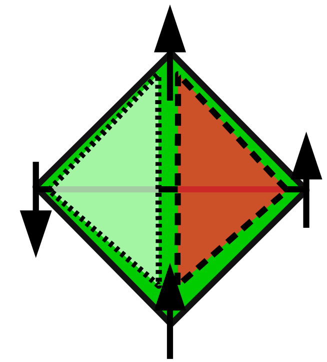

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- Spin language: $$H = \sum_{C} h(C) \quad \text{with} \quad h(C_{ijk}) = (\sigma_z^i + \sigma_z^j + \sigma_z^k - 1)^2$$
  Is the ground state energy $E=0$ or $E>0$?

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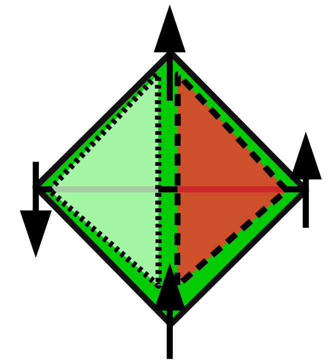

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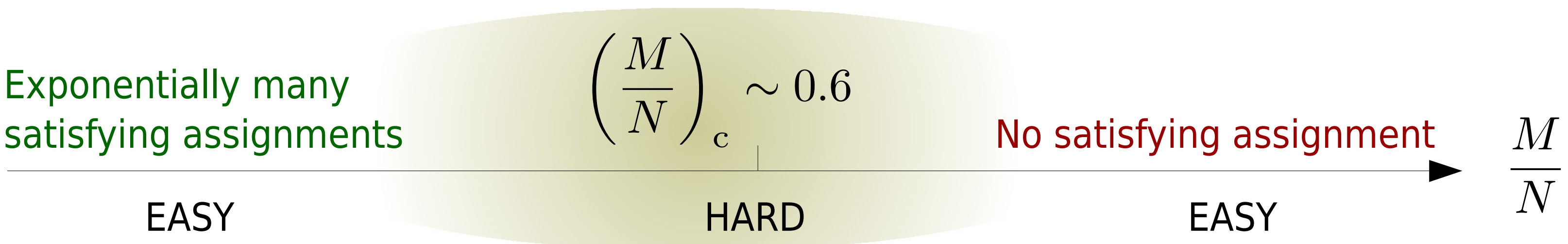


- Generate generically hard instances by constructing clauses with unique satisfying assignment (USA)

Annealing schedule: $\mathcal{H}(t) = (1-\frac{t}{T})H_\mathrm{t} + \frac{t}{T}H_\mathrm{p}$ with $H_\mathrm{t} = B\sum_i \sigma_x^i$

What is the probability of being in the ground state of H_p at time $t=T$?

E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, D. Preda
Science 292, 472 (2001)

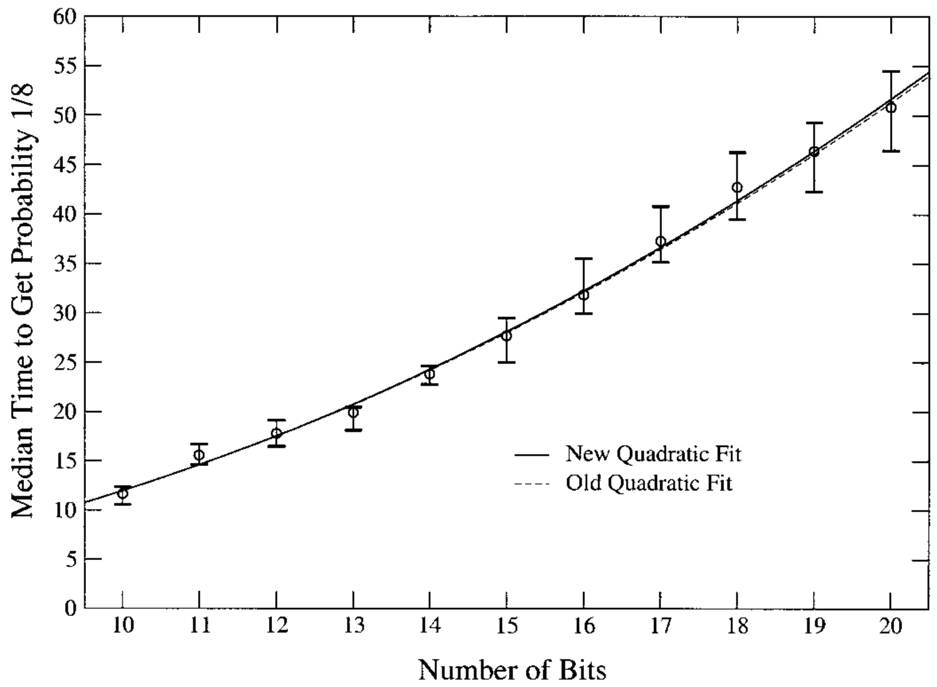
Numerical results:

- Success probability:

$$p = |\langle \text{GS of } H_\text{p} | \text{State at } t = T \rangle|^2$$

  For a given system size *N*, find runtime *T* such that *p* is fixed, e.g., *p*=1/8

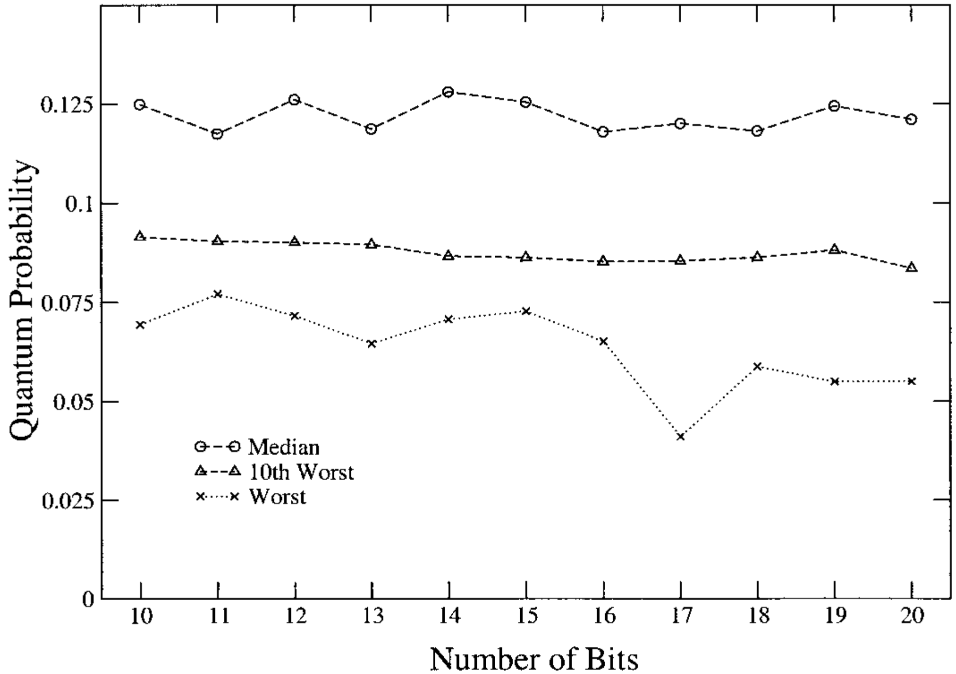
- *T* is found to scale polynomially with *N*



What about the worst instances?

- Averaged over the worst 10%, the probability remains relatively constant with *N*

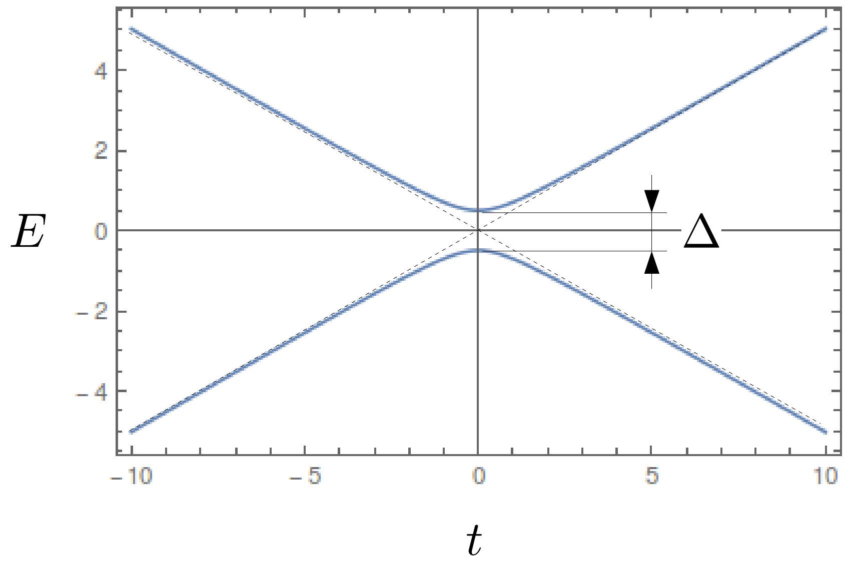
- Accessible system size too small to make strong claims regarding scalability



E. Farhi *et al.*,
Science 292, 472 (2001)

Adiabatic theorem and Landau-Zener problem

Spin-1/2 particle in a time-dependent magnetic field: $H(t) = \frac{1}{2}\gamma t \sigma_z + \frac{1}{2}\Delta\sigma_x$

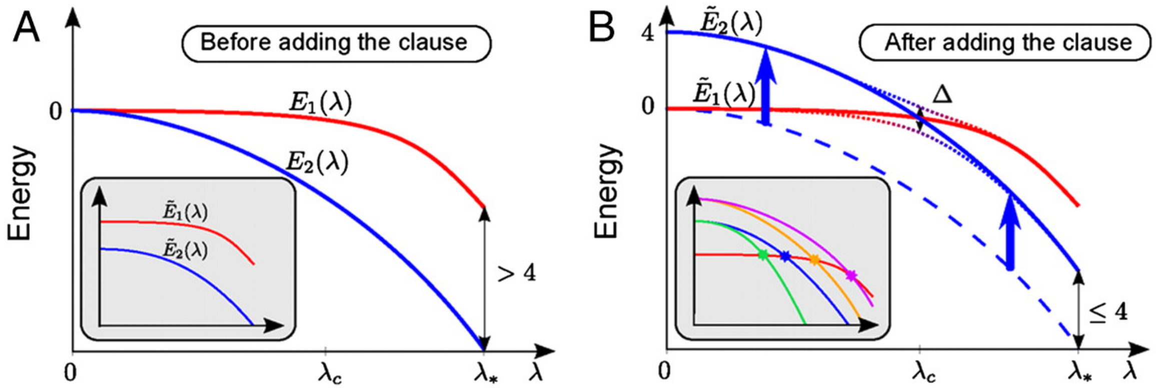


- Time-dependent Schroedinger equation: $i\hbar \begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \gamma t & \Delta \\ \Delta & -\gamma t \end{pmatrix} \cdot \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$
- Leads to one second-order differential equation which can be solved exactly
- Probability of excitation during the evolution: $P_{\mathrm{LZ}} = \mathrm{e}^{-\pi\Gamma^2/2}, \quad \Gamma \equiv \frac{\Delta}{\sqrt{\hbar\gamma}}$
- Adiabatic condition: $\Gamma \gg 1 \Rightarrow \Delta^2 \gg \hbar\gamma$
- The annealing time has to scale as $T \sim \frac{1}{\Delta^2}$

Anderson localization makes adiabatic quantum optimization fail

Boris Altshuler, Hari Krovi, and Jérémie Roland

PNAS July 13, 2010. 107 (28) 12446-12450; https://doi.org/10.1073/pnas.1002116107



Perturbative expansion:

$$E(\lambda, \mathbf{s}) = E_{\mathrm{cost}}(\mathbf{s}) + \sum_{m=1}^{\infty} \lambda^{2m} F^{(m)}$$

*m*th order connects configurations which are separated by up to *m* spin flips

$$E_1(\lambda) - E_2(\lambda) = \sum_{m=2}^{\infty} \lambda^{2m} [F_1^{(m)} - F_2^{(m)}]$$

First-order correction is the same for all configurations

$$N \gg 1: \quad F_i^{(m)} \cong \sum_{k=1}^{N} r_k, \quad r_k = \mathcal{O}(1)$$

For large N, the coefficients $F^{(m)}$ behave like sum of random numbers

$$\Rightarrow \langle F_1^{(m)} - F_2^{(m)} \rangle = 0, \quad \langle (F_1^{(m)} - F_2^{(m)})^2 \rangle = \mathcal{O}(N)$$

$$\Rightarrow |E_1(\lambda) - E_2(\lambda)| \cong \sqrt{N} \sum_m \lambda^{2m} \sim \sqrt{N}\lambda^4 \overset{!}{\sim} 4 \quad \Rightarrow \quad \lambda_{\mathrm{crossing}} \sim N^{-1/8}$$

leading order: *m*=2

$$\Delta \sim \lambda_{\mathrm{crossing}}^{m_{\mathrm{f}}} \sim \exp\{[-f(M/N)N/8] \ln(N/N_0)\}$$

$m_f \sim f(M/N)\, N$
"spin-flip-distance"

Can we avoid exponentially small gaps?

- Modify cost function?
  Exponentially small gaps have also been seen for other cost functions of similar complexity
  [T. Joerg, F. Krzakala, J. Kurchan, A.C. Maggs, PRL 101, 147204 (2008)]

- Modify the initial Hamiltonian?

  Standard protocol: $\mathcal{H}(t) = (1-\gamma) H_{\rm t} + \gamma H_{\rm p} \quad \text{with} \quad H_{\rm t} = B \sum_i \sigma_x^i$

  Modified protocol: $H_{\rm t} = B \sum_i \mu_i \sigma_x^i$

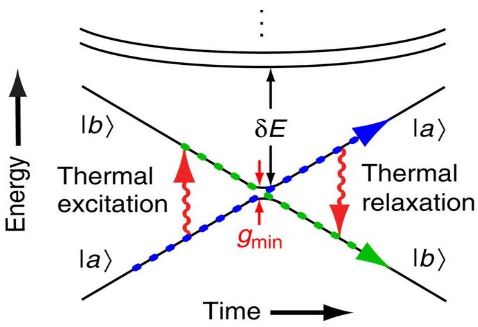
  [E. Farhi, J. Goldstone, D. Gosset, S. Gutmann, H. Meyer, P. Shor, Quant. Inf. Comp. 11, 181 (2011)]

- Make the Hamiltonian non-stoquastic?
  Stoquastic: No sign problem. All off-diagonal elements (in z-basis) are positive (e.g. transverse Ising)

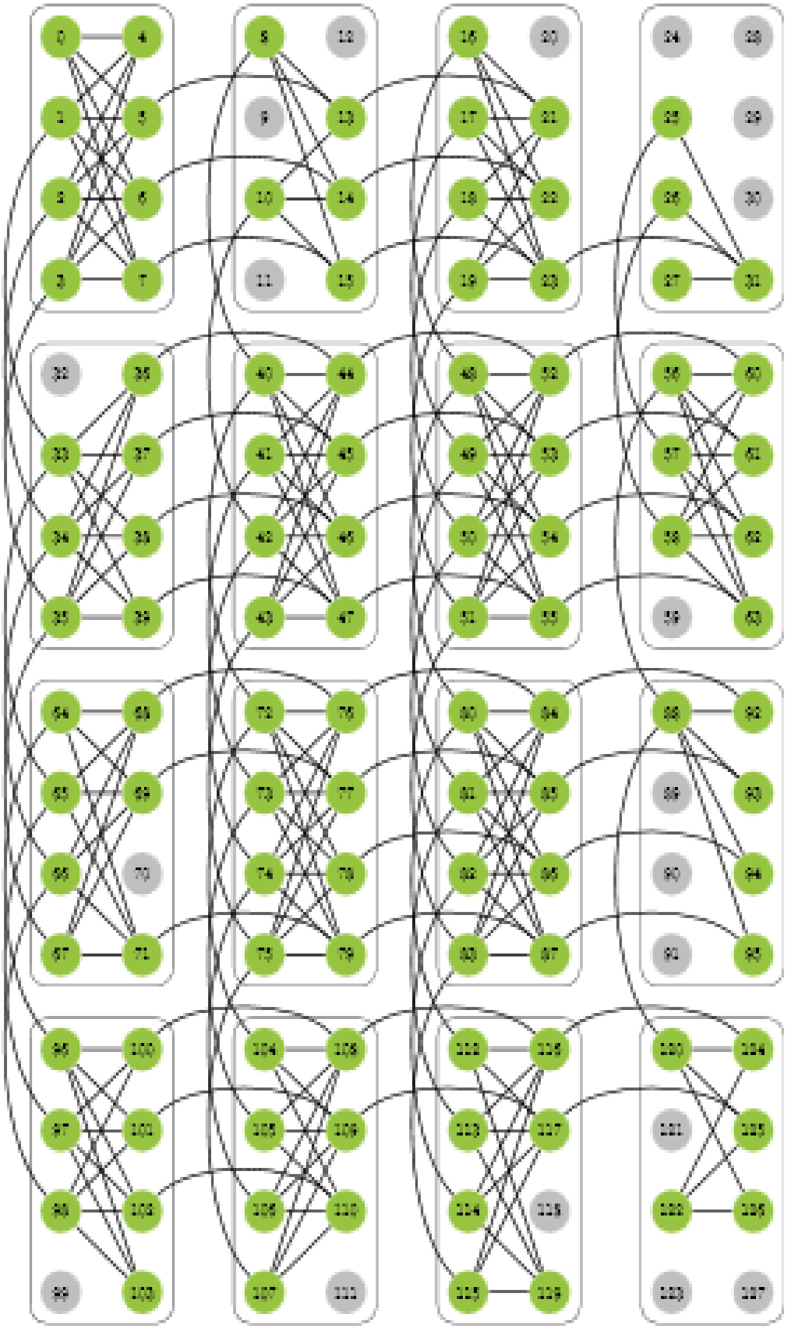
  Introducing terms which render the Hamiltonian non-stoquastic can turn 1st order phase transitions to 2nd order transitions:
  Example: $\mathcal{H}(\gamma, \lambda) = \gamma H_{\rm p} + (1-\gamma) H_{\rm t} + \gamma(1-\lambda) H_{\rm XX}$

  [H. Nishimori and K. Takada Front. ICT 4:2 (2017)]

- Thermal assisted quantum annealing?
  [N. Dickson et al., Nat. Commun. 4, 1903 (2013)]



Experimental realization: D-wave



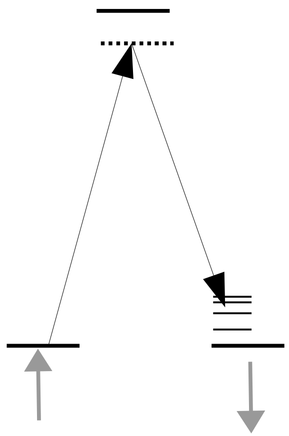
- Up to 2048 superconducting flux qubits
- Programmable Josephson couplings in blocks of 8 qubits and between blocks (chimera geometry)
- Individual longitudinal fields and global transverse field
- $$\mathcal{H}(t) = A(t)\left(\sum_{\langle i,j\rangle} J_{ij}\sigma_z^i\sigma_z^j + \sum h_i \sigma_z^i\right) + B(t)\sum_i \sigma_x^i$$
- Does it work? – Maybe!

  Output matches expectations
  [S. Boixo et al., Nature Phys. 10, 218 (2014)]

  No clear sign of quantum speedup
  [T. Ronnow et al., Science Science 345, 420 (2014)]

Alternative platforms: Trapped ions

- Raman spin-flip transition coupled to different phonon modes *m*
- Second-order: effective Ising model $H = -\sum_m \sum_{ij} \frac{\hbar \Omega^{(i)} \eta_m^{(i)} \Omega^{(j)} \eta_m^{(j)}}{\delta_m} \sigma_z^{(i)} \sigma_z^{(j)}$
- Close to a resonance we get a Mattis model: $H = -\mathrm{sign}(\delta) \sum_{ij} \xi^{(i)} \xi^{(j)} \sigma_z^{(i)} \sigma_z^{(j)}$



Energy: $E(s_1, \cdots, s_N) = -\mathrm{sign}(\delta) \left(\sum_{s_i=\uparrow} \xi^{(i)} - \sum_{s_i=\downarrow} \xi^{(i)} \right)^2$

Ferromagnetic coupling (δ>0):
Two ground states, all spins either aligned or anti-aligned with the parameter ξ_i

Antiferromagnetic coupling (δ<0):
Many ground states are possible. Hamiltonian represents the “number partitioning” problem

- Can we anneal the ions into the solution of this NP-hard problem?

  Exact diagonalization for 6 ions and phonons:
  Solution obtained on experimentally feasible time scales

  Semiclassical simulation for 22 ions and phonons:
  Annealing time scales as $\tau \propto N^4$

2	6	7	9	12	13	17	20
2	6	7	9	12	13	17	20
2	6	7	9	12	13	17	20

2+9+12+20 -6 -7-13 -17 = 0
6+17+20 -2 -7 -9 -12-13 = 0

T. Grass, D. Raventos, B. Julia-Diaz, C. Gogolin, M. Lewenstein
Nat. Commun. 7, 11524 (2016)

1) Simulated annealing:

- Cooling a problem to its solution
- Example: traveling salesperson problem

2) Quantum annealing:

- Quantum time evolution to the solution
- Examples: Spin models, exact Cover
- Adiabatic theorem, limitations, and workarounds
- Physical implementations (D-Wave, ions)

3) Phases of a computational problem:

Statistical physics analysis applied to the number partitioning problem

PHYSICAL REVIEW LETTERS

VOLUME 81 | 16 NOVEMBER 1998 | NUMBER 20

Phase Transition in the Number Partitioning Problem

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(Received 6 July 1998)

[T. Joerg, F. Krzakala, J. Kurchan, A.C. Maggs, PRL 101, 147204 (2008)]

ELSEVIER

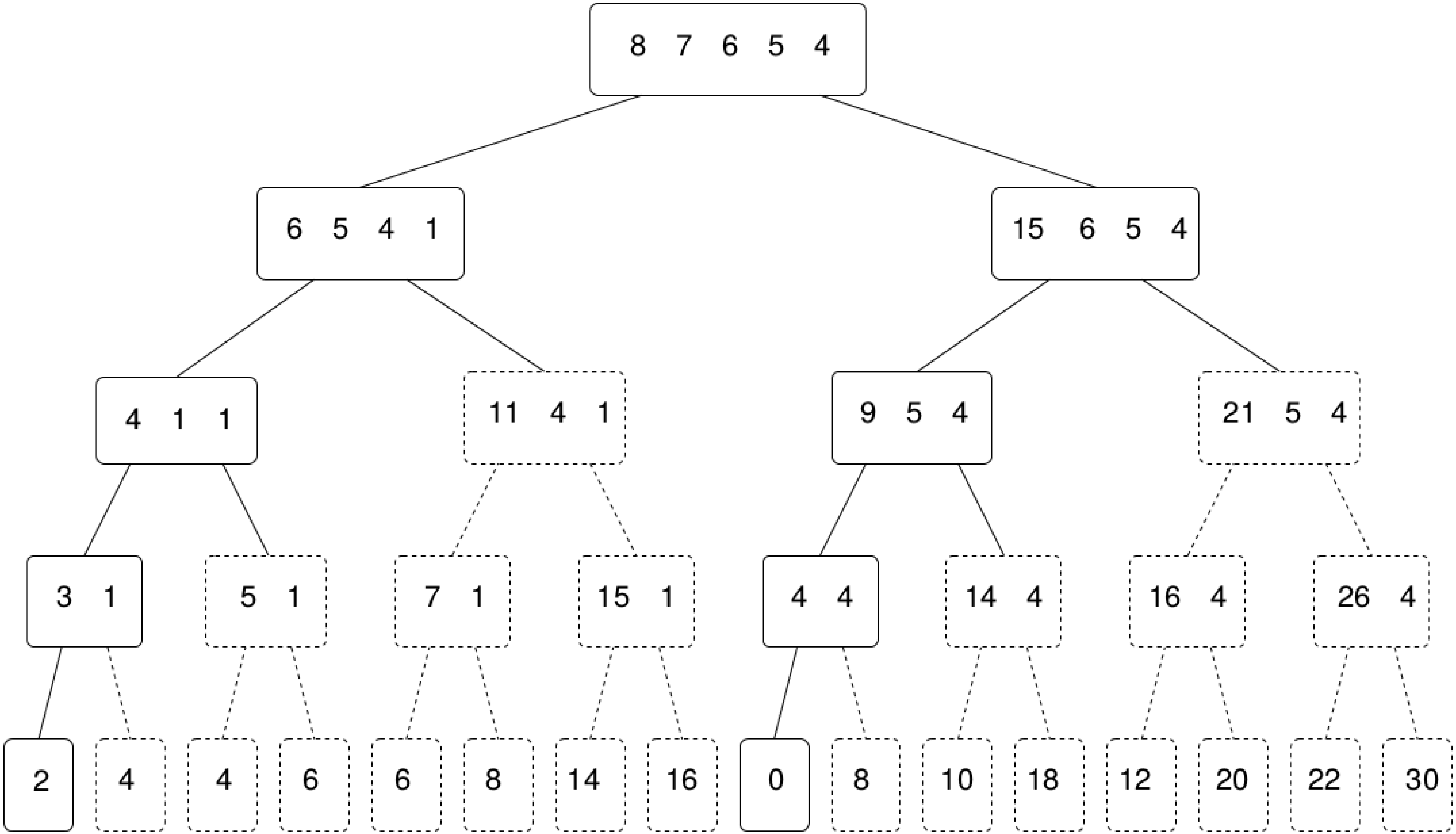
Theoretical Computer Science

Volume 265, Issues 1–2, 28 August 2001, Pages 79-108

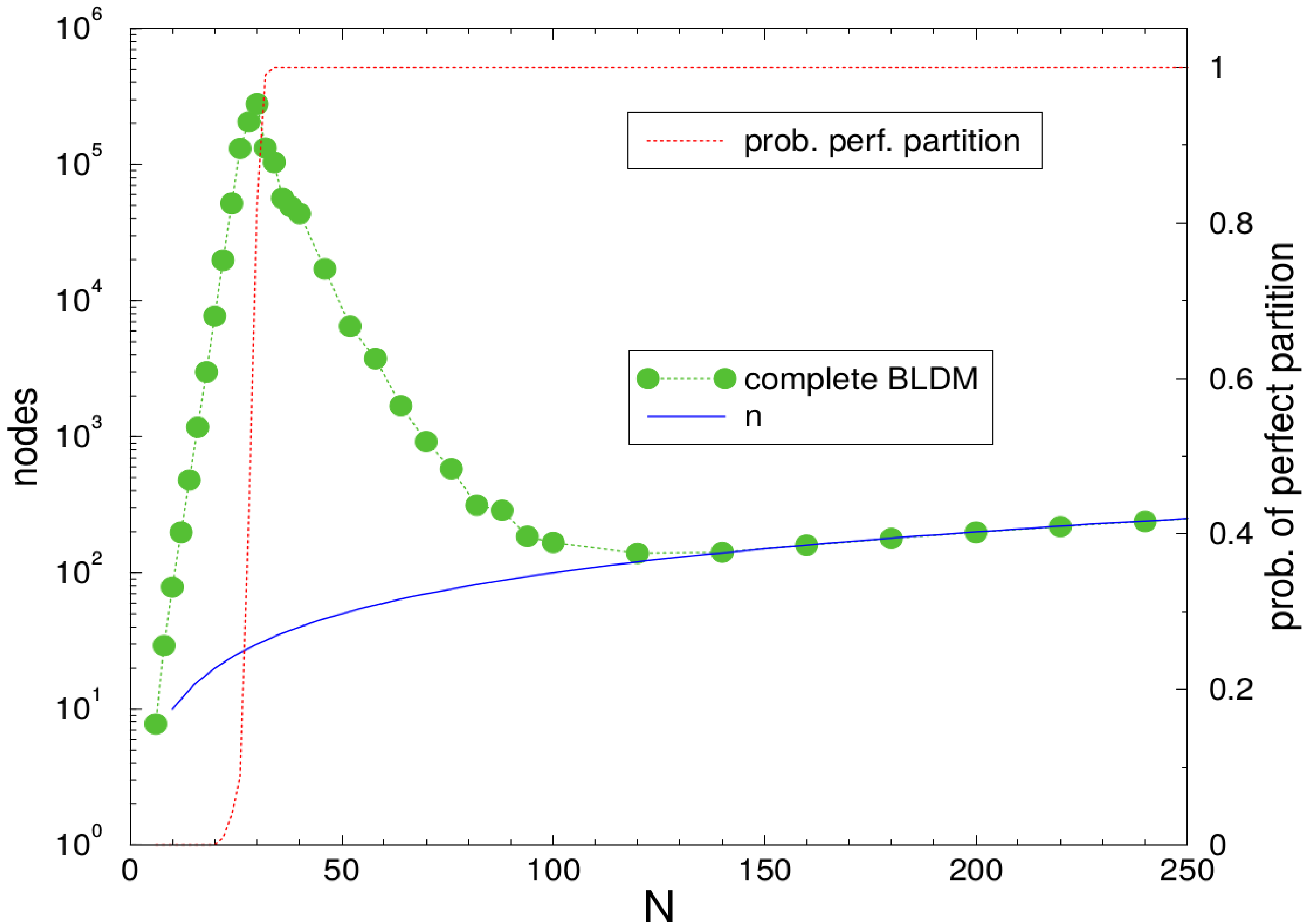
A physicist's approach to number partitioning

Stephan Mertens

Classical algorithm for number partitioning



Complexity of the problem



Statistical physics analysis of the problem

Cost function: $$E = |\sum_{j=1}^{N} a_j s_j|$$

Statistical physics analysis of the problem

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Statistical physics analysis of the problem

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Partition function: $$Z = \sum_{\{s_j\}} e^{-\frac{1}{T}|\sum_j a_j s_j|}$$

Without the absolute value, life would be easy:

$$\begin{aligned}
\sum_{\{s_j\}} e^{-\frac{1}{T}\sum_j a_j s_j} &= \sum_{\{s_j\}} \prod_{j=1}^{N} e^{-\frac{1}{T} a_j s_j} \\
&= \sum_{s_1=\pm 1} e^{-\frac{1}{T} a_1 s_1} \cdot \sum_{s_2=\pm 1} e^{-\frac{1}{T} a_2 s_2} \cdot \ldots \cdot \sum_{s_N=\pm 1} e^{-\frac{1}{T} a_N s_N} \\
&= 2\cosh\frac{a_1}{T} \cdot 2\cosh\frac{a_2}{T} \cdot \ldots \cdot 2\cosh\frac{a_N}{T} \\
&= 2^N \prod_{j=1}^{N} \cosh\frac{a_j}{T}
\end{aligned}$$

Cost function: $$E = |\sum_{j=1}^{N} a_j s_j|$$

Partition function: $$Z = \sum_{\{s_j\}} e^{-\frac{1}{T}|\sum_j a_j s_j|}$$

Use Dirac function:
$$Z = \sum_{\{s_j\}} \int_{-\infty}^{\infty} dx\, e^{-|x|}\, \delta(x - \frac{1}{T}\sum_{j=1}^{N} a_j s_j)$$
$$\overset{\star}{=} \int_{-\infty}^{\infty} dx\, e^{-|x|} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\hat{x} e^{ix\hat{x}} \sum_{\{s_j\}} e^{-i\frac{\hat{x}}{T}\sum_j a_j s_j}$$
$$= 2^N \int_{-\infty}^{\infty} \frac{d\hat{x}}{2\pi} \prod_{j=1}^{N} \cos(\frac{a_j}{T}\hat{x}) \int_{-\infty}^{\infty} dx\, e^{-|x|+i\hat{x}x} \quad \star$$

$\star$ $$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\hat{x} e^{ix\hat{x}}$$

$\star$ $$\int_{-\infty}^{\infty} dx\, e^{-|x|+i\hat{x}x} = \frac{2}{1+\hat{x}^2}$$

Statistical physics analysis of the problem

$$y = \arctan \hat{x}$$

Rewriting it:

$$Z = 2^N \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dy}{\pi} e^{N\,G(y)} \quad \text{with} \quad G(y) = \frac{1}{N}\sum_{j=1}^{N} \ln\cos(\frac{a_j}{T}\tan(y))$$

$$= \left\langle \ln\cos(\frac{a}{T}\tan(y)) \right\rangle$$

Statistical physics analysis of the problem

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$$= \left\langle \ln\cos(\frac{a}{T}\tan(y)) \right\rangle$$

Laplace method / Steepest descent method / Saddle-point method:

$$\int e^{NG(y)}dx \approx e^{NG(y_0)} \int e^{-\frac{N}{2}G''(y_0)(y-y_0)^2} dy = e^{NG(y_0)} \sqrt{\frac{2\pi}{NG''(y_0)}}$$

$$y = \arctan \hat{x}$$

Rewriting it:

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Saddle points:

$$G'(y) = \left\langle \frac{a}{T} \tan\left(\frac{a}{T} \tan y\right) \right\rangle \cdot (1 + \tan^2 y) = 0$$

$$y_k = \arctan\left(\frac{\pi T}{\Delta a} k\right) \qquad k = 0, \pm 1, \pm 2, \ldots$$

Numbers are discrete!

$$G''(y_k) = \frac{\langle a^2 \rangle}{T^2} \left[1 + \left(\frac{\pi T}{\Delta a} \right)^2 k^2 \right]^2$$

Result:

$$Z \approx 2^N \sum_k \int_{-\infty}^{\infty} \frac{dy}{\pi} e^{-\frac{N}{2} G''(y_k) y^2} = 2^N \frac{\sqrt{2}}{\sqrt{\pi N}} \sum_k \frac{1}{\sqrt{G''(y_k)}}.$$

$$\overset{\bigstar}{=} 2^N \cdot \frac{\Delta a}{\sqrt{\frac{\pi}{2} N \langle a^2 \rangle}} \cdot \coth \frac{\Delta a}{T}$$

$$\bigstar \sum_{k=0,\pm 1,\ldots} \frac{1}{1+(xk)^2} = \frac{\pi}{x} \cdot \coth \frac{\pi}{x}$$

Statistical physics analysis of the problem

Partition Function:

$$Z = 2^N \cdot \frac{\Delta a}{\sqrt{\frac{\pi}{2} N \langle a^2 \rangle}} \cdot \coth \frac{\Delta a}{T}$$

Free energy:

$$F(T) = -TN \ln 2 + \frac{T}{2} \ln \frac{\pi N \langle a^2 \rangle}{2 \Delta a^2} - T \ln \coth \frac{\Delta a}{T}$$

Thermal energy:

$$\langle E \rangle_T = \frac{\Delta a}{\sinh \frac{\Delta a}{T} \cosh \frac{\Delta a}{T}}$$

$$\lim_{T \to 0} \langle E \rangle_T = 0$$

Statistical physics analysis of the problem

Partition Function:

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Free energy:

$$F(T) = -TN \ln 2 + \frac{T}{2} \ln \frac{\pi N \langle a^2 \rangle}{2 \Delta a^2} - T \ln \coth \frac{\Delta a}{T}$$

Entropy:

$$S = N(\kappa_c - \kappa) \ln 2 + \tilde{S}(\frac{\Delta a}{2T}),$$

with

$$\kappa_c = 1 - \frac{\ln \left(\frac{\pi}{6} N \right)}{N \, 2 \ln 2}$$

$$\kappa = \frac{\ln \frac{3}{\Delta a^2} \langle a^2 \rangle}{N \, 2 \ln 2}$$

$$\tilde{S}(\frac{\Delta a}{T}) = \ln \coth \frac{\Delta a}{T} + \frac{\Delta a}{T} \frac{\coth^2 \frac{\Delta a}{T} - 1}{\coth \frac{\Delta a}{T}}$$

Statistical physics analysis of the problem

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$\kappa < \kappa_c$:

- extensive entropy
- exponentially many solutions
- "easy" phase

$\kappa > \kappa_c$:

- negative entropy?
- not at finite temperature
- There is no absolute zero → Energy will remain finite!
- "hard" phase

Statistical physics analysis of the problem

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$\kappa > \kappa_c$:

- negative entropy?
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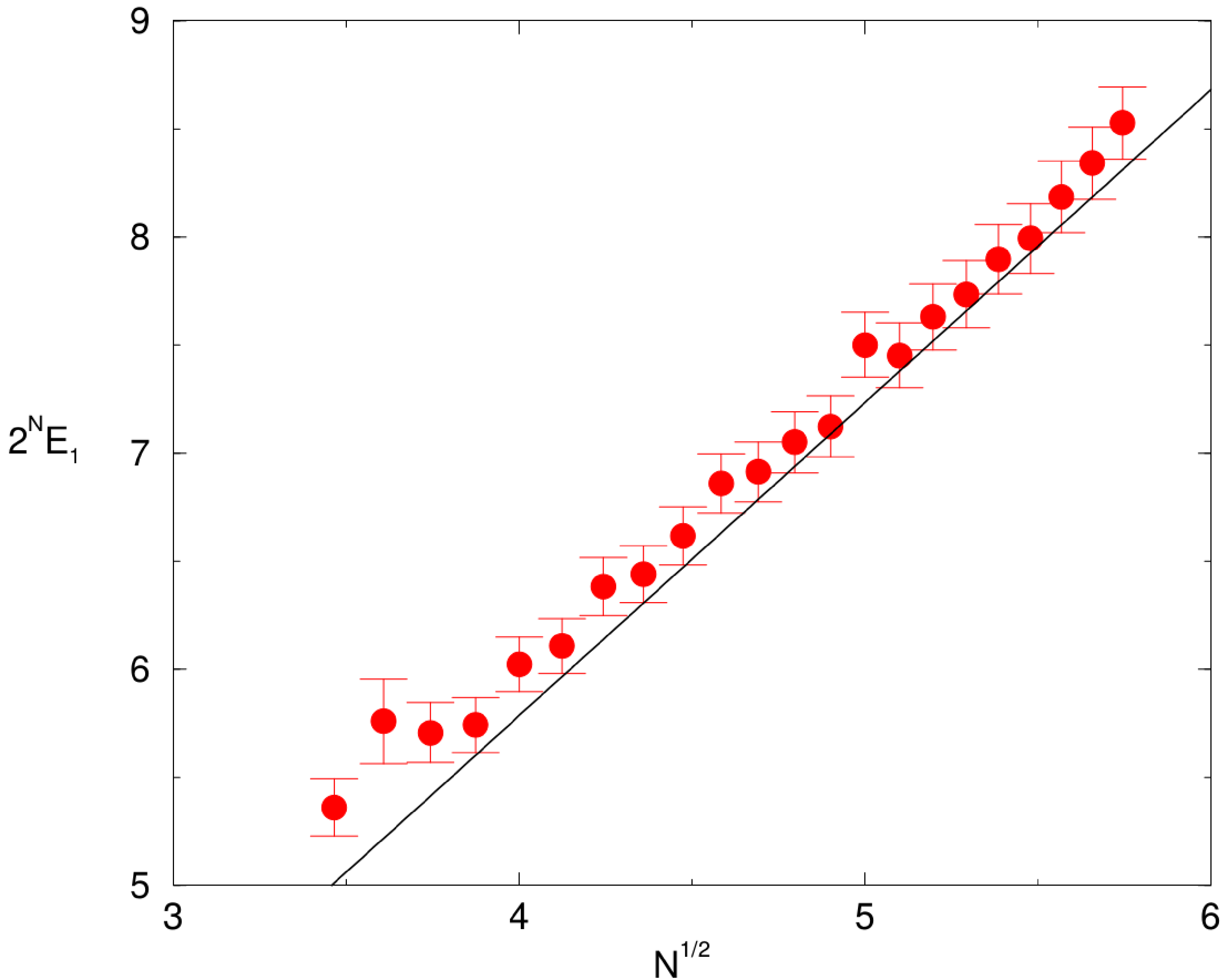
- There is no absolute zero → Energy will remain finite!
- "hard" phase

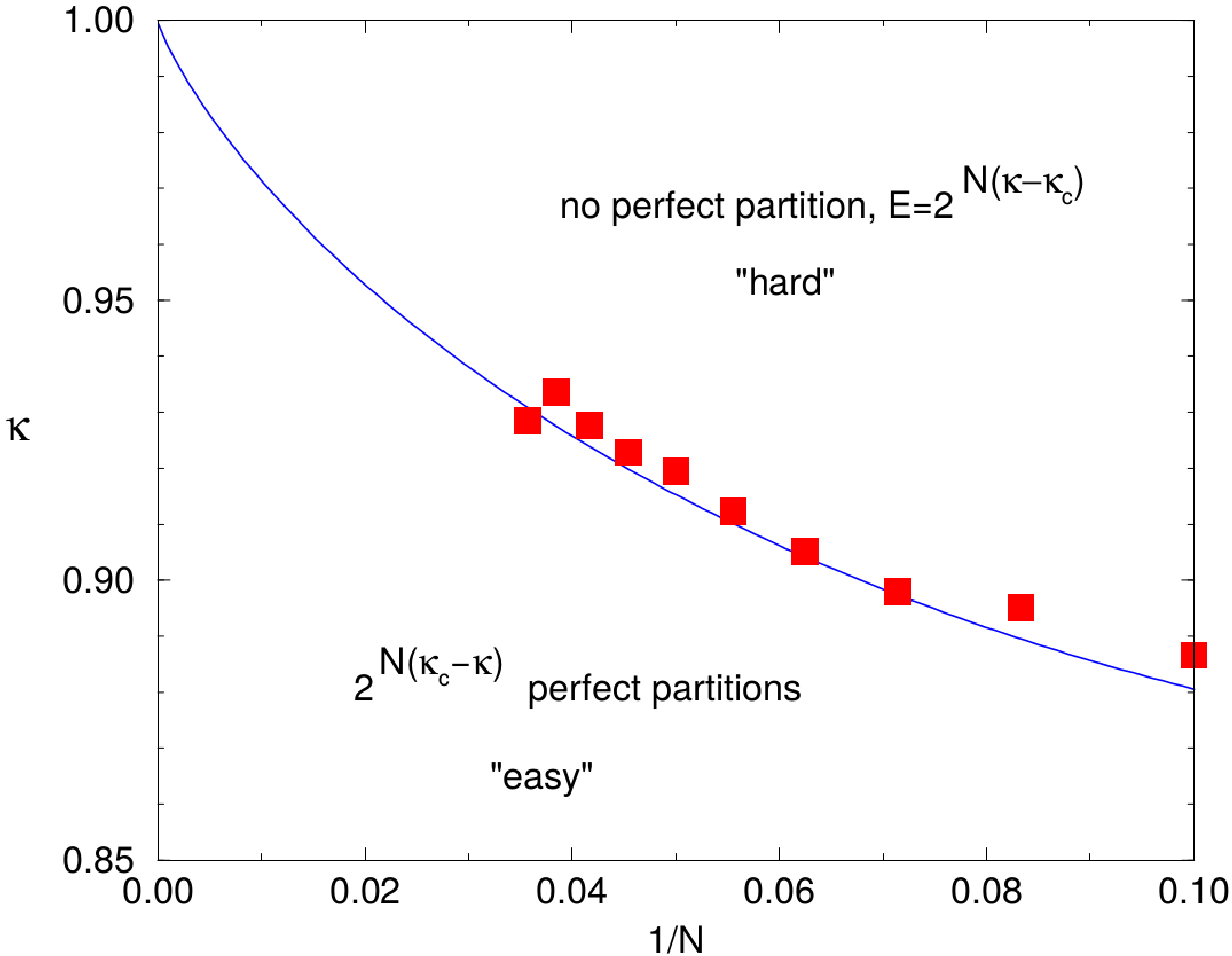
Minimum temperature and thermal energy when $\kappa > \kappa_c$:

$$T_0 = 2\Delta a\, 2^{N(\kappa-\kappa_c)} = \sqrt{2\pi N\langle a^2\rangle}\; 2^{-N}$$

$$\langle E_1\rangle = T_0 = \sqrt{2\pi N\langle a^2\rangle}\; 2^{-N}$$

Comparison with numerical results





1.00
0.95
0.90
0.85
κ
0.00
0.02
0.04
0.06
0.08
0.10
1/N
no perfect partition, $E=2^{N(\kappa-\kappa_c)}$
"hard"
$2^{N(\kappa_c-\kappa)}$ perfect partitions
"easy"

1) Simulated annealing:

- Simple sampling algorithm (Metropolis)
- Slow changes in control parameter ("temperature")

2) Quantum annealing:

- Adiabatic quantum time evolution
- Control parameter: transverse field
- Fails when small gaps occur along the annealing path

3) Analysis of the number partitioning problem:

"Entropy" becomes negative → phase transition to "hard" phase. "Absolute" zero at finite temperature/energy.